

Test 4

Section I, Part A

Allotted Time: 62 Minutes

Number of Questions: 29 Questions

Calculator Utilization: The use of a calculator is not permitted for this part of the test.

Instructions: Work through each of the problems, using the space provided to jot down calculations or notes. Carefully examine the answer choices for each question and determine which is the most accurate. Once you've selected the best answer, fill in the matching oval on the answer sheet. Keep in mind that anything written in this booklet won't be graded. Don't get stuck on any single question for too long. In this test, unless stated otherwise, assume that the domain of a function f includes all real values of x that result in $f(x)$ being a real number:

1. $\lim_{x \rightarrow 0} \frac{\sqrt{1+3x}-1}{x} =$

- a. $\frac{3}{2}$
- b. $\frac{1}{3}$
- c. $-\frac{3}{2}$
- d. $\frac{1}{2}$

2. If $f(x) = 4x^3 - 5e^x + 2 \sin x$, what is $f'(x)$?

- a. $12x^2 - 5e^x + 2 \cos x$
- b. $12x^2 - 5e^x + 2 \sin x$
- c. $12x^2 - 5e^x - 2 \cos x$
- d. $12x^2 + 5e^x + 2 \cos x$

3. The derivative of a function f is

$$f'(x) = (x - 2)^2(x + 1).$$

Which of the following statements is true?

- a. f has a local maximum at $x = -1$ and a local minimum at $x = 2$.
 - b. f has local minima at both $x = -1$ and $x = 2$.
 - c. f has a local minimum at $x = -1$ and no extremum at $x = 2$.
 - d. f has a local maximum at $x = 2$ and no extremum at $x = -1$.
4. The curve is given by

$$x^2 + xy + y^2 = 7.$$

What is $\frac{dy}{dx}$ at the point $(1, 2)$?

- a. $-\frac{2}{5}$
- b. $-\frac{4}{5}$
- c. $\frac{4}{5}$
- d. $\frac{2}{5}$

5. Let

$$G(x) = \int_2^x (t^2 - 1) dt.$$

What is $G'(3)$?

- a. 6
- b. 7
- c. 8
- d. 9

6. Let

$$f(x) = \begin{cases} kx + 4, & x < 2 \\ x^2 + 1, & x \geq 2 \end{cases}$$

What value of k makes f continuous at $x = 2$?

- a. 0
- b. 1
- c. $\frac{1}{2}$
- d. $\frac{1}{4}$

7. Find the area of the region bounded by $y = 9 - x^2$ and the x -axis.

- a. 12
- b. 18
- c. 27
- d. 36

8. Let $f(x) = x^2 \ln x$ for $x > 0$. What is the equation of the tangent line to f at $x = 1$?

- a. $y = 2x - 2$
- b. $y = x - 1$
- c. $y = 2x - 1$
- d. $y = x$

9. Suppose

$$f''(x) = x(x - 4)(x + 2).$$

At which x -value(s) does f have a point of inflection?

- a. $x = -2$ only
- b. $x = 0$ only
- c. $x = 4$ only
- d. $x = -2, 0$, and 4

10. A graph of f consists of line segments from $(0, 0)$ to $(2, 4)$ to $(6, 4)$ to $(8, 0)$. What is

$$\int_0^8 f(x) \, dx?$$

- a. 20
 - b. 24
 - c. 28
 - d. 32
11. A spherical balloon is being inflated so its radius increases at a constant rate of 2 cm/s. How fast is the volume increasing when the radius is 6 cm?

- a. $96\pi \frac{\text{cm}^3}{\text{s}}$
- b. $144\pi \frac{\text{cm}^3}{\text{s}}$
- c. $288\pi \frac{\text{cm}^3}{\text{s}}$
- d. $432\pi \frac{\text{cm}^3}{\text{s}}$

12. If $y = e^{\sin(5x)}$, then $\frac{dy}{dx} =$

- a. $5e^{\sin(5x)} \cos(5x)$
- b. $e^{\sin(5x)} \cos(5x)$
- c. $5e^{\cos(5x)} \sin(5x)$
- d. $e^{\sin(5x)} \sin(5x)$

13. Let $f(x) = x^2 + 2x$ on $[-1, 3]$. Find the value of c that satisfies the conclusion of the Mean Value Theorem.

- a. 0
- b. 1
- c. 2
- d. -1

14. Which integral is represented by

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(2 + \frac{3i}{n}\right)^2 \cdot \frac{3}{n} ?$$

- a. $\int_0^3 (2 + x)^2 dx$
- b. $\int_0^1 (2 + 3x)^2 dx$
- c. $\int_2^5 (2 + x)^2 dx$
- d. $\int_0^3 (2 + 3x)^2 dx$

15. Let

$$f(x) = \begin{cases} x + 1, & x < 0 \\ x^2 + 1, & x \geq 0 \end{cases}$$

Which statement is true?

- a. $\lim_{x \rightarrow 0} f(x) = 1$ and f is continuous at $x = 0$.
- b. $\lim_{x \rightarrow 0} f(x) = 0$ and f is continuous at $x = 0$.
- c. $\lim_{x \rightarrow 0^-} f(x) = 0$ and $\lim_{x \rightarrow 0^+} f(x) = 1$.
- d. $\lim_{x \rightarrow 0^-} f(x) = 1$ and $\lim_{x \rightarrow 0^+} f(x) = 1$, but f is not continuous at $x = 0$.

16. Find the area of the region enclosed by $y = x + 2$ and $y = x^2$.

- a. $\frac{9}{2}$
- b. $\frac{11}{6}$
- c. $\frac{7}{3}$
- d. $\frac{5}{2}$

17. For the differential equation $\frac{dy}{dx} = y - x$, at which point is the slope of the solution curve equal to 0?

- a. $(0, 1)$
- b. $(2, 0)$
- c. $(1, 1)$
- d. $(3, 1)$

18. If $h(x) = \ln(3x^2)$, then $h'(x) =$

- a. $\frac{6x}{3x^2}$
- b. $\frac{1}{3x^2}$
- c. $\frac{2}{x}$
- d. $\frac{1}{x}$

19. Evaluate

$$\int_0^2 (x+1)\sqrt{x^2+2x+5} \, dx.$$

- a. $5\sqrt{13} - 5\sqrt{5}$
- b. $\frac{1}{3}(13^{3/2} - 5^{3/2})$
- c. $\frac{1}{2}(13^{3/2} - 5^{3/2})$
- d. $\frac{2}{3}(13^{3/2} - 5^{3/2})$

20. Let $f(x) = x^3 - 3x$ on $[-2, 2]$. What is the absolute maximum value of f on the interval?

- a. 2
- b. 0
- c. 4
- d. 6

21. A particle moves along a line with velocity $v(t) = t^2 - 4t + 3$ and acceleration $a(t) = 2t - 4$, for $0 \leq t \leq 5$. At what time(s) is the particle moving to the right and slowing down?

- a. $0 < t < 1$
- b. $1 < t < 2$
- c. $2 < t < 3$
- d. $3 < t < 5$

22. If $f(4) = 7$ and $f'(4) = -2$, what is $(f^{-1})'(7)$?

- a. 2
- b. -2
- c. $\frac{1}{2}$
- d. $-\frac{1}{2}$

23. A car's velocity is $v(t) = 3t^2 - 6t$ meters per second for $0 \leq t \leq 4$. What is the displacement of the car on $[0, 4]$?

- a. -32 m
- b. 0 m
- c. 16 m
- d. 32 m

24. The graph of $f'(x)$ is described as follows:

- $f'(x) > 0$ for $x < -2$
- $f'(-2) = 0$
- $f'(x) < 0$ for $-2 < x < 1$
- $f'(1) = 0$
- $f'(x) > 0$ for $x > 1$

Which statement is true?

- a. f has a local minimum at $x = -2$ and a local maximum at $x = 1$.
- b. f has a local maximum at $x = -2$ and a local minimum at $x = 1$.
- c. f has local minima at both $x = -2$ and $x = 1$.
- d. f has local maxima at both $x = -2$ and $x = 1$.

25. Find the average value of $f(x) = \sin(x)$ on $[0, \pi]$.

- a. 0
- b. $\frac{1}{\pi}$
- c. $\frac{\pi}{2}$
- d. $\frac{2}{\pi}$

26. Let $f(x) = x^2 - 3x$. Evaluate

$$\lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}.$$

- a. 0
- b. 3
- c. 6
- d. 9

27. Suppose

$$\int_1^5 f(x) \, dx = 10 \quad \text{and} \quad \int_1^3 f(x) \, dx = 4.$$

What is $\int_5^3 f(x) \, dx$?

- a. 14
- b. -14
- c. 6
- d. -6

28. Evaluate

$$\lim_{x \rightarrow 0} \frac{\sin(7x)}{x}.$$

- a. 7
- b. 1
- c. $\frac{1}{7}$
- d. -7

29. Which function satisfies $\frac{dy}{dx} = 2y$ and $y(0) = 3$?

- a. $y = 3e^{2x}$
- b. $y = 3e^x$
- c. $y = 2e^{3x}$
- d. $y = 6e^x$

Section I, Part B

Allotted Time: 38 Minutes

Number of Questions: 13 Questions

Calculator Utilization: The use of a calculator is permitted for the questions in this part of the test.

Instructions: For each problem below, solve using the space provided for scratch work. Review the format of the answer options, select the best choice, and mark the corresponding oval on your answer sheet. Note that any work written in the test booklet will not receive credit, so focus on completing the answer sheet. Avoid spending excessive time on any single question. The exact numerical answer may not always be included in the answer choices. In such cases, select the option that most closely approximates the exact value. Unless stated otherwise, the domain of a function is assumed to include all real numbers x for which $f(x)$ produces a real number.

30. Values of a continuous function f are shown below.

x	0	1	2	3	4	5
$f(x)$	2.0	2.6	3.1	3.8	4.5	5.0

Using the trapezoidal rule with 5 subintervals, approximate $\int_0^5 f(x) \, dx$.

- a. 15.8
- b. 16.2
- c. 17.5
- d. 17.8

31. The curve is defined implicitly by

$$x^2 + xy + 2y^2 = 10.$$

At the point $(1.5, 1.8)$, approximate $\frac{dy}{dx}$.

- a. -0.74
- b. -0.55
- c. 0.55
- d. 0.74

32. Let $g(x) = x \sin x - \ln(x + 2)$ on $[0, 6]$. Use a calculator to determine the absolute maximum value of g on $[0, 6]$.

- a. 0.29
- b. 0.38
- c. 0.44
- d. 0.71

33. A function $s(t)$ is measured at the times shown below:

t	3.8	3.9	4.0	4.1	4.2
$s(t)$	12.4	13.1	13.9	14.8	15.8

Estimate $s'(4)$.

- a. 8.5
- b. 7.0
- c. 9.0
- d. 8.0

34. Values of a function $h(x)$ near $x = 2$ are shown.

x	1.99	1.999	2.001	2.01
$h(x)$	5.97	5.997	6.003	6.03

What is the best estimate of $\lim_{x \rightarrow 2} h(x)$?

- a. 5.97
- b. 6.03
- c. The limit does not exist
- d. 6.00

35. The region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$ is revolved about the x -axis. What is the volume of the resulting solid?

- a. $\frac{32\pi}{3}$
- b. 8π
- c. 16π
- d. $\frac{64\pi}{3}$

36. A function y satisfies

$$\frac{dy}{dx} = \frac{2x}{y+1}, \quad y(0) = 2.$$

Use the differential equation and initial condition to find $y(3)$. (Round to the nearest hundredth.)

- a. 3.61
- b. 3.92
- c. 4.05
- d. 4.20

37. Let $f(x) = e^{-x} \sin(x^2)$. Use a calculator to approximate $f'(1)$.

- a. -0.31
- b. 0.02
- c. 0.09
- d. 0.28

38. Let

$$A(x) = \int_0^x (e^{-t^2} + t^2) dt.$$

Use a calculator to approximate $A(2)$.

- a. 3.55
- b. 2.67
- c. 3.10
- d. 4.02

39. Let $f(x) = \ln(x + 2) + \cos(2x) - 0.2x$. Use a calculator to approximate the value of x in the interval $[1, 2]$ where $f'(x) = 0$.
- a. 1.12
 - b. 1.36
 - c. 1.74
 - d. 1.55
40. Find the average value of $f(x) = e^{-x^2}$ on $[0, 2]$.
- a. 0.22
 - b. 0.44
 - c. 0.56
 - d. 0.12
41. A population is $P = f(T)$ and temperature is $T = g(t)$. Suppose $f'(20) = 1.6$ and $g'(5) = -0.7$. If $g(5) = 20$, what is $\frac{dP}{dt}$ at $t = 5$?
- a. -1.12
 - b. -0.91
 - c. 0.91
 - d. 1.12
42. A graphing calculator shows the graph of a rate function $r(t)$. The signed area under $r(t)$ from $t = 0$ to $t = 6$ is 12, and the signed area from $t = 6$ to $t = 10$ is -5 . What is $\int_0^{10} r(t) dt$?
- a. 12
 - b. -5
 - c. 17
 - d. 7

Section II, Part A

Allotted Time: 30 Minutes

Number of Questions: 2 Questions

Calculator Utilization: The use of a calculator is permitted for the questions in this part of the test.

Instructions: Providing your set up for these problems is essential, as partial credit may be granted. If you include decimal approximations, ensure they are accurate to three decimal places.

1. A company's daily profit (in thousands of dollars) is modelled

$$P(x) = x^3 - 12x^2 + 36x + 10 \ln(x + 1)$$

for $0 \leq x \leq 10$, where x is the number of hundreds of units produced.

- Find $P'(x)$.
 - Use a calculator to find all critical points of P on the open interval $(0, 10)$. Give your answers to three decimal places.
 - Determine the value of x on $[0, 10]$ at which $P(x)$ is maximized. Justify your answer.
 - The concavity of P is related to changes in how profit is increasing or decreasing. Find all x -values in $(0, 10)$ where P has a point of inflection. Justify your answer.
2. A pond initially contains 1,200 gallons of water at time $t = 0$. Water is being added to the pond at a rate of $W(t)$ gallons per hour. Selected values of $W(t)$ are shown in the table below.

t (hours)	0	1	3	5	6	8	10
$W(t)$ (gal/hr)	85	102	118	110	96	74	60

- a. Use the trapezoidal rule with the data in the table to approximate

$$\int_0^{10} W(t) \, dt.$$

- b. Interpret the meaning of

$$\int_0^{10} W(t) \, dt$$

in the context of the problem. Include units.

- Find the average rate at which water is added to the pond over the interval $0 \leq t \leq 10$. Include units.
- Use your answer from part (a) to approximate the amount of water in the pond at time $t = 10$.

Section II, Part B

Allotted Time: 60 Minutes

Number of Questions: 4 Questions

Calculator Utilization: The use of a calculator is not permitted for this part of the test.

3. Let f be the piecewise function

$$f(x) = \begin{cases} ax^2 + bx + 3, & x \leq 2 \\ 4\sqrt{x+2} + c, & x > 2 \end{cases}$$

where a , b , and c are all constants.

- Find the value of c in terms of a and b so that f is continuous at $x = 2$.
- Find a relationship between a and b so that f is differentiable at $x = 2$.
- Assuming f is continuous and differentiable at $x = 2$, write the equation of the tangent line to f at $x = 2$.
- Assume that f is differentiable at $x = 2$. If $a = 1$, find the values of b and c .

4. Let

$$f(x) = x^3 - 6x^2 + 9x + 2.$$

- Find $f'(x)$ and determine all critical values of f .
- On which intervals is f increasing? On which intervals is f decreasing? Justify your answer.
- Find all x -values where f has a point of inflection. Justify your answer.
- Write an equation for the tangent line to the graph of f at $x = 2$.

5. Let f be a continuous function defined on $[0, 9]$. The graph of f is made of the following pieces:
- From $x = 0$ to $x = 3$, f is a line segment from $(0, 2)$ to $(3, -1)$.
 - From $x = 3$ to $x = 7$, f is the lower semicircle with center $(5, -1)$ and radius 2.
 - From $x = 7$ to $x = 9$, f is a line segment from $(7, -1)$ to $(9, 3)$.

Define

$$g(x) = \int_0^x f(t) dt.$$

- Find $g(3)$.
 - Find $g'(8)$.
 - On which interval(s) in $[0, 9]$ is g decreasing? Justify your answer.
 - Find the absolute minimum value of $g(x)$ on $[0, 9]$. Justify your answer.
6. A function $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = x(y - 1).$$

- Show that $y = 1$ is a solution to the differential equation.
- Find the general solution to the differential equation.
- Find the particular solution that satisfies $f(0) = 3$.
- For the particular solution from part (c), determine the interval(s) of x for which the solution is decreasing. Justify your answer.

Solutions

Section I, Part A

1. A	2. A	3. C	4. B	5. C
6. C	7. D	8. B	9. D	10. B
11. C	12. A	13. B	14. A	15. A
16. A	17. C	18. C	19. B	20. A
21. A	22. D	23. C	24. B	25. D
26. B	27. D	28. A	29. A	

1. Simply multiply by the conjugate, then evaluate.

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sqrt{1+3x} - 1}{x} &\cdot \frac{\sqrt{1+3x} + 1}{\sqrt{1+3x} + 1} \\
 &= \lim_{x \rightarrow 0} \frac{1 + 3x - 1}{x(\sqrt{1+3x} + 1)} \\
 &= \lim_{x \rightarrow 0} \frac{3}{\sqrt{1+3x} + 1} = \frac{3}{2}
 \end{aligned}$$

2. Differentiate term-by-term:

$$\frac{d}{dx}(4x^3) = 12x^2, \quad \frac{d}{dx}(-5e^x) = -5e^x, \quad \frac{d}{dx}(2 \sin x) = 2 \cos x$$

3. f' changes from negative to positive at $x = -1$ (local minimum) and does not change sign at $x = 2$ (no extremum).
4. Differentiate to get

$$\begin{aligned}
 2x + \left(y + x \frac{dy}{dx}\right) + 2y \frac{dy}{dx} &= 0 \\
 \frac{dy}{dx}(x + 2y) &= -2x - y \\
 \frac{dy}{dx} &= -\frac{2x + y}{x + 2y}
 \end{aligned}$$

Then plug in $(1, 2)$.

$$\frac{dy}{dx} = -\frac{2(1) + (2)}{(1) + 2(2)} = -\frac{4}{5}$$

5. By the Fundamental Theorem of Calculus, $G'(x) = x^2 - 1$, so $G'(3) = 9 - 1 = 8$.

6. Set $2k + 4 = 2^2 + 1 = 5 \Rightarrow k = \frac{1}{2}$.

7. The intercepts are at $x = \pm 3$. So, the area would be given by:

$$\int_{-3}^3 (9 - x^2) dx = \left[9x - \frac{1}{3}x^3 \right]_{-3}^3 = 36$$

8. $f(1) = 0$ and $f'(x) = 2x \ln x + x$. So, $f'(1) = 1$. Therefore, the tangent line equation is:

$$y = f'(1)(x - 1) + f(1) = 1(x - 1) + 0 = x - 1$$

9. f'' changes sign at each zero $-2, 0$, and 4 . Therefore, all three zeros are points of inflection of f .

10. Area = left triangle + rectangle + right triangle = $4 + 16 + 4 = 24$

11. $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi(36)(2) = 288\pi$

12. Apply the Chain Rule to find the derivative:

$$e^{\sin(5x)} \cdot (\cos(5x) \cdot 5).$$

13. First, we need to find average slope:

$$\frac{f(3) - f(-1)}{3 - (-1)} = \frac{15 - (-1)}{4} = \frac{16}{4} = 4.$$

Since $f'(x) = 2x + 2$, set $2c + 2 = 4$, then solve for c .

$$2c + 2 = 4 \Rightarrow 2c = 2 \Rightarrow c = 1.$$

14. $\Delta x = \frac{b-a}{n} = \frac{3}{n}$ over $[0, 3]$, and $x_i = \frac{3i}{n}$, so the term in the Riemann Sum is $(2 + x_i)^2$. Therefore, the corresponding function to integrate is $(2 + x)^2$.

15. Left limit = $0 + 1 = 1$. Right limit = $0^2 + 1 = 1$. Also, $f(0) = 1$. Therefore, f is continuous at $x = 0$.

16. First, we need to find where these functions intersect: $x^2 = x + 2 \Rightarrow x = -1, 2$. On $(-1, 2)$, the line is above the parabola.

$$\int_{-1}^2 [(x + 2) - x^2] dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^2 = \frac{9}{2}$$

17. Slope 0 occurs where $\frac{dy}{dx} = 0 \Rightarrow y - x = 0 \Rightarrow y = x$. Only $(1, 1)$ fits this requirement.

18. By applying Chain Rule and Power Rule:

$$h'(x) = \frac{1}{3x^2} \cdot 6x = \frac{6x}{3x^2} = \frac{2}{x}$$

19. Let $u = x^2 + 2x + 5 \Rightarrow du = 2(x + 1)dx$. Then the given integral becomes

$$\frac{1}{2} \int_{u=5}^{13} u^{1/2} du = \frac{1}{2} \cdot \frac{2}{3} [u^{3/2}] \Big|_5^{13} = \frac{1}{3} (13^{3/2} - 5^{3/2})$$

20. First, find the critical values of f on the given interval $[-2, 2]$:

$$f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1.$$

Then, evaluate f at the critical values and endpoints of the interval.

$$f(-2) = -2, \quad f(-1) = 2, \quad f(1) = -2, \quad f(2) = 2.$$

So, the maximum value on the interval is 2.

21. $v(t) = (t - 1)(t - 3)$, so $v > 0$ on $(0, 1) \cup (3, 5)$. Also, $a(t) = 2(t - 2)$, so $a < 0$ on $(0, 2)$.

Moving to the right and slowing down means $v > 0$ and $a < 0$, which occurs only on $(0, 1)$.

22. Applying the derivative of an inverse formula:

$$(f^{-1})'(7) = \frac{1}{f'(f^{-1}(7))} = \frac{1}{f'(4)} = \frac{1}{-2} = -\frac{1}{2}$$

23. To find displacement, simply evaluate the definite integral of velocity over the interval.

$$\int_0^4 (3t^2 - 6t) dt = [t^3 - 3t^2] \Big|_0^4 = 64 - 48 = 16$$

24. Slope changing from positive to negative at $x = -2$ gives a local maximum and changing from negative to positive at $x = 1$ gives a local minimum.

25. Simply using the Average Value formula:

$$\frac{1}{\pi} \int_0^\pi \sin x \, dx = \frac{1}{\pi} (2) = \frac{2}{\pi}$$

26. This is the limit definition of $f'(3)$; since $f'(x) = 2x - 3$, $f'(3) = 3$.

27. Using definite integral properties:

$$\int_3^5 f(x) \, dx = \int_1^5 f(x) \, dx - \int_1^3 f(x) \, dx = 10 - 4 = 6.$$

So, switching the bounds gives

$$\int_5^3 f(x) \, dx = -6.$$

28. By L'Hospital's Rule:

$$\lim_{x \rightarrow 0} \frac{\sin(7x)}{x} = \lim_{x \rightarrow 0} \frac{\cos(7x) \cdot 7}{1} = \frac{1 \cdot 7}{1} = 7$$

29. This is a separable differential equation with general solution $y = Ce^{2x}$. Using the initial condition $y(0) = 3 \Rightarrow C = 3$. Alternatively, you can test the given multiple-choice options by finding their

derivatives and plugging them into the given differential equation. For $y = 3e^{2x}$, $\frac{dy}{dx} = 2(3e^{2x})$ and $y(0) = 3e^{2(0)} = 3$.

Section I, Part B

30. C	31. B	32. C	33. A	34. D
35. B	36. D	37. C	38. A	39. D
40. B	41. A	42. D		

30. Using the trapezoidal rule formula with $\Delta x = 1$:

$$T = \frac{1}{2} [f(0) + 2(f(1) + f(2) + f(3) + f(4)) + f(5)] = 17.5$$

31. First, differentiate implicitly. Then solve for $\frac{dy}{dx}$, and plug in the point (1.5, 1.8):

$$2x + y + x \frac{dy}{dx} + 4y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x + y}{x + 4y} \approx -0.55$$

32. Use the calculator's maximum feature on $[0, 6]$; the maximum value is approximately 0.44.

33. Calculate with symmetric difference quotient:

$$s'(4) \approx \frac{s(4.1) - s(3.9)}{4.1 - 3.9} = \frac{14.8 - 13.1}{0.2} = 8.5.$$

34. The values of $h(x)$ approach 6 from both sides of $x = 2$.

35. By the disk method:

$$V = \pi \int_0^4 (\sqrt{x})^2 dx = \pi \int_0^4 x dx = \pi \cdot \frac{16}{2} = 8\pi.$$

36. Separate and integrate:

$$(y + 1) dy = 2x dx \Rightarrow \frac{y^2}{2} + y = x^2 + C.$$

Use $y(0) = 2$:

$$\frac{4}{2} + 2 = 0 + C \Rightarrow 4 = C.$$

At $x = 3$:

$$\begin{aligned} \frac{y^2}{2} + y &= 13 \\ y^2 + 2y - 26 &= 0. \end{aligned}$$

Solve this equation using your calculator to get: $y \approx 4.196$ or -6.196 (which is not one of the answer choices), so $y(3) \approx 4.20$.

37. Use the nDeriv function in your calculator at $x = 1$ to find $f'(1) \approx 0.09$.

38. Numerical integration gives $A(2) \approx 3.55$.

39. First find the derivative:

$$f'(x) = \frac{1}{x+2} - 2 \sin(2x) - 0.2.$$

Then by graphing, we can use the calculator to solve $f'(x) = 0$ on $[1, 2]$. The solution is about 1.55.

40. Average value $= \frac{1}{2} \int_0^2 e^{-x^2} dx$. Plugging this into your calculator gives approximately 0.44.

41. By the Chain Rule:

$$\frac{dP}{dt} = f'(T) \cdot \frac{dT}{dt} = 1.6(-0.7) = -1.12.$$

42. Add the signed areas: $12 + (-5) = 7$.

Section II, Part A

1. (a) Using the Power Rule and derivative of natural log:

$$P'(x) = 3x^2 - 24x + 36 + \frac{10}{x+1}$$

- (b) Critical points occur where $P'(x) = 0$ (and where $P'(x)$ is undefined, but that doesn't happen on the given interval). Use a calculator to solve

$$3x^2 - 24x + 36 + \frac{10}{x+1} = 0$$

on $(0, 10)$.

$$x \approx 2.273, 5.875$$

- (c) To find the absolute maximum on $[0, 10]$, compare $P(x)$ at:

- Endpoints $x = 0, 10$
- Critical points from part (b)

Using your calculator:

- $P(0) = 0$
- $P(2.273) \approx 43.430$
- $P(5.875) \approx 19.371$
- $P(10) \approx 183.979$

The greatest of these values occurs at $x = 10$, so profit is maximized at $x = 10$.

- (d) Inflection points occur where $P''(x) = 0$ and concavity changes. First find:

$$P''(x) = 6x - 24 - \frac{10}{(x+1)^2}.$$

Use a calculator to solve:

$$6x - 24 - \frac{10}{(x+1)^2} = 0$$

On $(0, 10)$, $x \approx 4.065$ is the point of inflection. Check that $P''(x)$ changes sign around that x -value to confirm that it is an inflection point.

2. (a) Because the time intervals are not all the same length, us trapezoids on each interval separately:

$$\begin{aligned} \int_0^{10} W(t) dt &\approx \frac{1}{2}(1)(85 + 102) + \frac{1}{2}(2)(102 + 118) + \frac{1}{2}(2)(118 + 110) \\ &\quad + \frac{1}{2}(1)(110 + 96) + \frac{1}{2}(2)(96 + 74) + \frac{1}{2}(2)(74 + 60). \\ &= 93.5 + 220 + 228 + 103 + 170 + 134 = 948.5. \end{aligned}$$

$$\int_0^{10} W(t) dt \approx 948.5 \text{ gallons}$$

(b) $\int_0^{10} W(t) dt \approx 948.5$ gallons is the amount of water added to the pond in the first 10 hours.

(c) Average value:

$$W_{ave} = \frac{1}{10 - 0} \int_0^{10} W(t) dt$$

Using the approximation from part (a):

$$W_{ave} \approx \frac{948.5}{10} = 94.85 \text{ gal/hr.}$$

(d) The pond initially contains 1,200 gallons. From part (a), about 948.5 gallons are added.

$$1200 + 948.5 = 2148.5.$$

The pond contains approximately 2,148.5 gallons at $t = 10$.

Section II, Part B

3. (a) Continuity requires $\lim_{x \rightarrow 2^-} f(x) = f(2) = \lim_{x \rightarrow 2^+} f(x)$.

Left-hand limit and value at $x = 2$:

$$\lim_{x \rightarrow 2^-} f(x) = f(2) = a(2)^2 + b(2) + 3 = 4a + 2b + 3.$$

Right-hand limit:

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (4\sqrt{x+2} + c) = 4\sqrt{4} + c = 8 + c.$$

Set equal,

$$\begin{aligned} 4a + 2b + 3 &= 8 + c \\ c &= 4a + 2b - 5 \end{aligned}$$

- (b) Differentiability requires matching derivatives from the left and right at $x = 2$.

Left derivative ($x \leq 2$):

$$\begin{aligned} f'(x) &= 2ax + b \\ f'_-(2) &= 4a + b. \end{aligned}$$

Right derivative ($x > 2$):

$$\begin{aligned} f'(x) &= 4 \cdot \frac{1}{2\sqrt{x+2}} = \frac{2}{\sqrt{x+2}} \\ f'_+(2) &= \frac{2}{\sqrt{4}} = 1. \end{aligned}$$

Set equal:

$$4a + b = 1.$$

- (c) If f is differentiable at $x = 2$, then slope is $f'(2) = 1$ (from part (b)), and the point is $(2, f(2))$ where $f(2) = 4a + 2b + 3$ (from part (a)).

Using point-slope form:

$$\begin{aligned} y - (4a + 2b + 3) &= 1(x - 2) \\ y &= x + 4a + 2b + 1. \end{aligned}$$

- (d) Using part (b), we know $4a + b = 1$. Therefore, if $a = 1$:

$$\begin{aligned} 4(1) + b &= 1 \\ b &= -3. \end{aligned}$$

Now use the continuity condition from part (a):

$$c = 4a + 2b - 5.$$

Substitute $a = 1$ and $b = -3$:

$$c = 4(1) + 2(-3) - 5 = 4 - 6 - 5 = -7.$$

$$b = -3, \quad c = -7$$

4. (a) First, find the derivative:

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3).$$

Critical values occur where $f'(x) = 0$: $x = 1$ and $x = 3$.

$$f'(x) = 3(x - 1)(x - 3), \text{ critical values: } x = 1, 3$$

(b) Use the sign of $f'(x) = 3(x - 1)(x - 3)$. This can be shown with a gradient diagram.

x	0	1	2	3	4
Sign of $f'(x)$	+	0	−	0	+
Direction of $f(x)$					

f is increasing on $(-\infty, 1) \cup (3, \infty)$, and decreasing on $(1, 3)$.

(c) First, find the second derivative:

$$f''(x) = 6x - 12 = 6(x - 2).$$

Inflection points may occur where $f''(x) = 0$: $x = 2$. Since $f''(x)$ changes sign from negative (when $x < 2$) to positive (when $x > 2$), f has an inflection point at $x = 2$.

(d) First, find the point:

$$f(2) = 2^3 - 6(2)^2 + 9(2) + 2$$

$$f(2) = 8 - 24 + 18 + 2 = 4.$$

Now, find the slope:

$$f'(2) = 3(2 - 1)(2 - 3) = 3(1)(-1) = -3.$$

So, the tangent line has slope -3 and passes through $(2, 4)$:

$$y - 4 = -3(x - 2).$$

or

$$y = -3x + 10.$$

5. (a) Compute $g(3)$:

$$g(3) = \int_0^3 f(t) \, dt.$$

From $x = 0$ to $x = 3$, the graph is a line segment from $(0, 2)$ to $(3, -1)$. The signed area is the area above the x -axis minus the area below the x -axis.

The line has slope -1 , so it crosses the x -axis at $x = 2$.

Area above the x -axis:

$$\frac{1}{2}(2)(2) = 2$$

Area below the x -axis:

$$\frac{1}{2}(1)(1) = \frac{1}{2}$$

So

$$g(3) = 2 - \frac{1}{2} = \frac{3}{2}.$$

$$g(3) = \frac{3}{2}$$

- (b) By the Fundamental Theorem of Calculus,

$$g'(x) = f(x).$$

On $[7, 9]$, f is the line from $(7, -1)$ to $(9, 3)$ with slope $\frac{3-(-1)}{9-7} = 2$.

So, its equation is

$$y - (-1) = 2(x - 7)$$

$$f(x) = 2x - 15.$$

Therefore,

$$g'(8) = f(8) = 2(8) - 15 = 1.$$

$$g'(8) = 1$$

- (c) g is decreasing where $g'(x) \leq 0$, i.e., where $f(x) \leq 0$. From part (a), the first line segment crosses the x -axis at $x = 2$, so $f(x) < 0$ on $(2, 3]$.

The lower semicircle from $x = 3$ to $x = 7$ is entirely below the x -axis, so $f(x) < 0$ on $[3, 7]$.

On the last line segment, $f(x) = 2x - 15$, so $f(x) < 0$ when

$$2x - 15 < 0 \Rightarrow x < \frac{15}{2}.$$

On $[7, 9]$, this gives $\left(7, \frac{15}{2}\right)$.

Combining these:

g is increasing on $\left(2, \frac{15}{2}\right)$

(d) An absolute maximum of g on a closed interval occurs at endpoints or where $g'(x) = 0$.

Since $g'(x) = f(x)$, candidates are $x = 0$, $x = 2$, $x = \frac{15}{2}$, and $x = 9$.

Compute $g(x)$ at these points:

- $g(0) = 0$
- $g(2) = \int_0^2 f(t) dt$

Area from $x = 0$ to 2 is a triangle with area $\frac{1}{2}(2)(2) = 2$.

- $g\left(\frac{15}{2}\right)$, use the accumulated area.

From $x = 0$ to 3:

$$g(3) = \frac{3}{2}.$$

From $x = 3$ to 7, the lower semicircle has radius 2, centered at $y = -1$. The signed area from 3 to 7 is the area of the rectangle below the x -axis plus the lower semicircle below $y = -1$:

$$-4 - \frac{1}{2}\pi(2^2) = -4 - 2\pi.$$

From $x = 7$ to $\frac{15}{2}$, the graph is below the x -axis and forms a triangle with base $\frac{1}{2}$ and height 1:

$$-\frac{1}{2}\left(\frac{1}{2}\right)(1) = -\frac{1}{4}.$$

Thus,

$$g\left(\frac{15}{2}\right) = \frac{3}{2} - 4 - 2\pi - \frac{1}{4} = -\frac{11}{4} - 2\pi.$$

- $g(9)$, from $x = 7$ to 9, the signed areas below and above the x -axis partially cancel:
 - From $x = 7$ to $\frac{15}{2}$: signed area $-\frac{1}{4}$
 - From $x = \frac{15}{2}$ to 9: triangle with base $\frac{3}{2}$, height 3:

$$\frac{1}{2}\left(\frac{3}{2}\right)(3) = \frac{9}{4}.$$

So

$$g(9) = g(7) - \frac{1}{4} + \frac{9}{4} = -\frac{5}{2} - 2\pi + 2 = -\frac{1}{2} - 2\pi.$$

Compare $g(0)$, $g(2)$, $g\left(\frac{15}{2}\right)$, and $g(9)$.

The smallest value is $g\left(\frac{15}{2}\right) = -\frac{11}{4} - 2\pi$.

The absolute minimum value of g on $[0, 9]$ is $-\frac{11}{4} - 2\pi$ (at $x = \frac{15}{2}$).

6. (a) If $y = 1$, we can take the derivative of both sides with respect to x so find $\frac{dy}{dx} = 0$. Plug these values into the differential equation to verify the solution.

$$\frac{dy}{dx} = x(y - 1)$$

$$0 = x((1) - 1)$$

$$0 = x(0)$$

$$0 = 0$$

(b) Separate the variables:

$$\frac{dy}{y - 1} = x \, dx$$

Integrate:

$$\int \frac{dy}{y - 1} = \int x \, dx$$

$$\ln |y - 1| = \frac{1}{2}x^2 + C_1$$

Isolate y :

$$y - 1 = e^{\frac{1}{2}x^2 + C_1} = Ce^{\frac{1}{2}x^2}$$

$$y(x) = Ce^{\frac{1}{2}x^2} + 1$$

(c) Plug in $x = 0, y = 3$:

$$3 = Ce^{\frac{1}{2}(0)^2} + 1$$

$$3 = Ce^0 + 1$$

$$2 = Ce^0$$

$$C = 2$$

So,

$$f(x) = 2e^{\frac{1}{2}x^2} + 1.$$

(d) The solution is decreasing where $\frac{dy}{dx} < 0$.

Using the differential equation:

$$\frac{dy}{dx} = x(y - 1).$$

For the particular solution, $y - 1 = 2e^{\frac{1}{2}x^2}$.

Since

$$2e^{\frac{1}{2}x^2} > 0$$

for all real x , the sign of $\frac{dy}{dx}$ depends only on x .

Therefore, $\frac{dy}{dx} < 0$ when $x < 0$.

The particular solution is decreasing on $(-\infty, 0)$.